GRUNSKY-NEHARI INEQUALITIES FOR A SUBCLASS OF BOUNDED UNIVALENT FUNCTIONS

BY

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Abstract. Let D_1 be the class of regular analytic functions $F(z)$ in the disc $U=\{z : |z|<1\}$ for which $F(0)>0$, $|F(z)|<1$, and $F(z)+F(\zeta)\neq 0$ for all z, $\zeta \in U$. Inequalities of the Grunsky-Nehari type are obtained for the univalent functions in D_1, the proof being based on the area method. By subordination it is shown univalency is unnecessary for certain special cases of the inequalities. Employing a correspondence between D_1 and the class S_1 of bounded univalent functions, the results can be reinterpreted to apply to this latter class.

1. **Introduction of the function classes.** In this section we provide motivation for the introduction of the function classes which we shall denote in the sequel by D, D_1, and $D(\beta)$. The univalent functions within these classes (which we indicate by a * superscript) have an interesting connection with the following classes of univalent functions defined on the unit disc $U=\{z : |z|<1\}$.

Class S. The functions $g(z)$ which are regular analytic and univalent in U, and whose Taylor series expansion about the origin is of the form

$$(1) \qquad g(z) = z+a_2z^2+\cdots+a_nz^n+\cdots.$$

Class S_1. The functions $f(z)$ which are regular analytic and univalent in U, have a Taylor series expansion about the origin of the form

$$(2) \qquad f(z) = b_1z+b_2z^2+\cdots+b_nz^n+\cdots, \qquad b_1 > 0,$$

and which are bounded (by unity):

$$(3) \qquad |f(z)| < 1.$$

Class $S(b_1)$. The functions $f(z)$ in S_1 which have a prescribed first derivative $f'(0)=b_1$ at the origin, $0<b_1\leq 1$.

It should be noted that if $f(z) \in S(b_1)$ then $b_1^{-1}f(z) \in S$. Under this renormalization S_1 can be considered to be a dense subset of S, and the coefficients $a_n \equiv b_1^{-1}b_n$ belong to the S function $b_1^{-1}f(z)=b_1^{-1}(b_1z+\cdots+b_nz^n+\cdots)$. Here dense refers to the topology of uniform convergence on the compact subsets of the disc U.

Let $g(z) \in S$ and suppose u is a complex number for which $g(z) \neq 1/u$. We can then form the function

(4) $$G(z) = \sqrt{(1 - ug(z))}, \qquad G(0) = 1,$$

which is again regular analytic and univalent in U. Moreover, the univalency of $g(z)$ assures us that if $G(z)$ takes a value w then it must omit its negative $-w$. This property is characteristic of a function class studied by S. A. Guelfer [3].

Class D (Guelfer). The functions $G(z)$ which are regular analytic in U, have a Taylor series expansion about the origin of the form

(5) $$G(z) = 1 + \alpha_1 z + \cdots + \alpha_n z^n + \cdots,$$

and which satisfy the property

(6) $$G(z) + G(\zeta) \neq 0$$

for all pairs of numbers $z, \zeta \in U$.

Guelfer proved a number of distortion and rotation theorems for D and $D^* \equiv \{G(z) \in D : G \text{ univalent}\}$, and also derived certain estimates for the coefficients α_n. Inverting the transformation (4) he obtained some of the well-known results for functions $g(z)$ in S which omit a value $1/u$. For the proofs Guelfer relied on the close relationship which exists between D and the class C of Bieberbach-Eilenberg functions (see [4]).

Class C (Bieberbach-Eilenberg). The functions $E(z)$ which are regular analytic in U, and with the properties that $E(0) = 0$ and

(7) $$E(z)E(\zeta) \neq 1$$

for all pairs $z, \zeta \in U$.

If $G(z) \in D$, we easily see

(8) $$E(z) = (1 - G(z))/(1 + G(z))$$

is a function in C. Thus results known for the class C can be utilized to obtain information about the functions of D.

If transformations (4) and (8) are combined, we obtain a correspondence between S and C^* which has been considered recently by Hummel and Schiffer [4]. By the area technique they derive for C^* functions some general inequalities which are equivalent (via the aforementioned correspondence) to the Garabedian-Schiffer inequalities [2] within the class S. The approach through Bieberbach-Eilenberg functions thus provides a proof of elementary character for the Garabedian-Schiffer inequalities, in distinction to the original derivation by variational means.

We shall now show how the above considerations can be modified for the case of bounded functions. Let $f(z) \in S_1$ and suppose

(9) $$f(z) \neq \beta^2 e^{-i\varphi}, \qquad 0 < \beta < 1.$$

The function

(10) $$F(z) = \sqrt{\left(\frac{\beta^2 - e^{i\varphi} f(z)}{1 - \beta^2 e^{i\varphi} f(z)} \right)}, \qquad F(0) = \beta,$$

is again regular analytic and univalent in U. Moreover, $F(z)$ has the Guelfer property (6) and is bounded: $|F(z)| < 1$. In this way we are prompted to define the following two additional function classes.

Class D_1. The functions $F(z)$ which are regular analytic in U and satisfy the following properties:

$$(11) \qquad\qquad F(0) > 0,$$

$$(12) \qquad\qquad |F(z)| < 1,$$

$$(13) \qquad\qquad F(z) + F(\zeta) \neq 0$$

for all pairs $z, \zeta \in U$.

Class $D(\beta)$. The functions $F(z)$ in D_1 which map the origin to a prescribed point $\beta \in (0, 1)$, so that

$$(14) \qquad\qquad F(0) = \beta.$$

Of course $\beta^{-1} F(z) \in D$ whenever $F(z) \in D(\beta)$, and D_1 is dense in D under this renormalization. More importantly it is to be observed that the transformation (10) may be inverted in a unique way when $F(z)$ is univalent. Indeed, let $F(z) \in D^*(\beta)$ and suppose

$$(15) \qquad F(z) = \beta + \beta_1 z + \cdots + \beta_n z^n + \cdots.$$

Then if $\alpha_1 = \beta^{-1} \beta_1 = -|\alpha_1| e^{i\varphi}$ we can form

$$(16) \qquad f(z) = e^{i\varphi}(\beta^2 - F^2(z))/(1 - \beta^2 F^2(z)).$$

The defining properties (12) and (14) assure us $f(z)$ is a bounded regular function which vanishes at the origin. To check univalency, an easy calculation shows that

$$(17) \qquad f(z) - f(\zeta) = -e^{-i\varphi} \frac{(F(z) - F(\zeta))(F(z) + F(\zeta))(1 - \beta^4)}{(1 - \beta^2 F^2(z))(1 - \beta^2 F^2(\zeta))}.$$

The univalency of $F(z)$, along with the Guelfer property (13), implies (17) vanishes only if $z = \zeta$. Finally we notice that

$$(18) \qquad f'(0) = (2\beta^2/(1 - \beta^4))|\alpha_1| > 0$$

and so we conclude $f(z) \in S(b_1)$, with $f(z) \neq \beta^2 e^{-i\varphi}$ and

$$(19) \qquad b_1 = (2\beta^2/(1 - \beta^4))|\alpha_1|.$$

In the next section we obtain a general area inequality for D_1^*. By means of the correspondence (16) we show how this provides an elementary derivation of inequalities recently obtained by Nehari [5] and DeTemple [1] for the class S_1. It is to be noted that although Nehari's proof is of the area type to be considered here, his formulation of the inequalities omits the important complementing role of the Guelfer functions D_1^*; in DeTemple on the other hand, the characteristic properties of the D_1^* class are emphasized, but the variational proof reverts attention again to the class S_1.

2. **An area theorem for univalent Guelfer functions.** For any $F(z) \in D^*(\beta)$, we may consider the following power series expansions:

$$\log \frac{F(z) - F(\zeta)}{z - \zeta} = \sum_{m,n=0}^{\infty} k_{mn} z^m \zeta^n,$$

$$-\log [F(z) + F(\zeta)] = \sum_{m,n=0}^{\infty} l_{mn} z^m \zeta^n,$$

(20)

$$\log [1 + F(z)\overline{F(\zeta)}] = \sum_{m,n=0}^{\infty} p_{mn} z^m \overline{\zeta}^n,$$

$$-\log [1 - F(z)\overline{F(\zeta)}] = \sum_{m,n=0}^{\infty} q_{mn} z^m \overline{\zeta}^n.$$

The properties of the class $D^*(\beta)$ guarantee the convergence of these series in the bicylinder $U \times U$. It should be observed that the coefficients k_{mn} and l_{mn} are symmetric in their indices, while p_{mn} and q_{mn} are Hermitian symmetric.

Next we define a sequence of rational functions $\Phi_n(t)$ by means of the generating function

(21) $$\log \frac{1 - t\beta}{1 - tF(\zeta)} = \sum_{n=1}^{\infty} \frac{1}{n} \Phi_n(t)\zeta^n.$$

The $\Phi_n(t)$ are closely related to the Faber polynomials (see Schiffer [7]). Indeed, if $F_n(s)$ is the nth Faber polynomial for $[F(z) - \beta]$, then

(22) $$\Phi_n(t) = F_n(t/(1 - \beta t)).$$

It follows that each $\Phi_n(t)$ has a pole only at $t = \beta^{-1}$.

We utilize the functions $\Phi_n(t)$ to expand the left sides of (20) in powers of ζ, $|\zeta|$ small, holding z fixed. Comparing equal powers of ζ, we deduce the identities

$$\sum_{m=0}^{\infty} k_{m0} z^m = \log \frac{F(z) - \beta}{z},$$

$$n \sum_{m=0}^{\infty} k_{mn} z^m = z^{-n} - \Phi_n\left[\frac{1}{F(z)}\right] \qquad (n \geq 1),$$

$$\sum_{m=0}^{\infty} l_{m0} z^m = -\log [\beta + F(z)],$$

(23)

$$n \sum_{m=0}^{\infty} l_{mn} z^m = \Phi_n\left[-\frac{1}{F(z)}\right] \qquad (n \geq 1),$$

$$\sum_{m=0}^{\infty} p_{m0} z^m = \log [1 + \beta F(z)],$$

$$n \sum_{m=0}^{\infty} p_{mn} z^m = -\overline{\Phi_n[-\overline{F(z)}]} \qquad (n \geq 1),$$

$$\sum_{m=0}^{\infty} q_{m0} z^m = -\log [1 - \beta F(z)],$$

$$n \sum_{m=0}^{\infty} q_{mn} z^m = \overline{\Phi_n[\overline{F(z)}]} \qquad (n \geq 1).$$

The identities (23) enable us to define a function $q(w)$ for which the inherently positive integral $\iint |q'(w)|^2 \, d\tau$ may be evaluated when taken over an appropriate domain. The basic idea is to consider simultaneously both $F(z) \in D^*(\beta)$ and its negative

$$(24) \qquad \hat{F}(z) = -F(z).$$

The ranges of $F(z)$ and $\hat{F}(z)$ are nonoverlapping subdomains of the unit disc $|w| < 1$. In particular, by choosing a ρ $(0 < \rho < 1)$ we may define two nonintersecting simple curves γ and $\hat{\gamma}$ inside $|w| < 1$ with the parametric representations

$$(25) \qquad w = F(\rho e^{i\theta}), \qquad w = \hat{F}(\rho e^{i\theta}).$$

Along with the unit circumference $|w| = 1$ they bound a triply-connected domain Δ_ρ which contains neither $\pm\beta$.

The function $q(w)$ is now defined as follows. Let x_0, y_0 be any two real numbers and let x_n, y_n $(n = 1, 2, \ldots, N)$ be any $2N$ complex numbers. Then let

$$
(26) \qquad
\begin{aligned}
q(w) = {}& x_0 \log \frac{w - \beta}{1 - \beta w} - y_0 \log \frac{w + \beta}{1 + \beta w} \\
& - \sum_{m=1}^{N} \left[\frac{x_m}{m} \Phi_m(w^{-1}) - \frac{\overline{x_m}}{m} \overline{\Phi_m(\overline{w})} \right] \\
& + \sum_{m=1}^{N} \left[\frac{y_m}{m} \Phi_m(-w^{-1}) - \frac{\overline{y_m}}{m} \overline{\Phi_m(-\overline{w})} \right].
\end{aligned}
$$

We observe that $q(w)$ is analytic in Δ_ρ and has there a single-valued real part in view of the realness of x_0 and y_0. Using the complex version of Green's identity, we can write

$$(27) \qquad 0 \leqq \iint_{\Delta_\rho} |q'(w)|^2 \, d\tau = \frac{1}{i} \oint_{\partial\Delta_\rho} \operatorname{Re}\{q(w)\} \, dq(w),$$

where $d\tau$ is the area element in Δ_ρ. From (26) we note that $\operatorname{Re}\{q(w)\} = 0$ on $|w| = 1$. Thus from the parametric representations (25) for the remaining two components γ and $\hat{\gamma}$ of the boundary $\partial\Delta_\rho$, we see (27) goes over to

$$(28) \qquad I_1 + I_2 \leqq 0,$$

where

$$
(29) \qquad
\begin{aligned}
I_1 &= \frac{1}{i} \int_0^{2\pi} \operatorname{Re}\{q[F(\rho e^{i\theta})]\} \frac{dq[F]}{d\theta} \, d\theta, \\
I_2 &= \frac{1}{i} \int_0^{2\pi} \operatorname{Re}\{q[\hat{F}(\rho e^{i\theta})]\} \frac{dq[\hat{F}]}{d\theta} \, d\theta.
\end{aligned}
$$

The identities (23) enable us to express these integrals in terms of the coefficients of the series (20). Indeed, we find after straightforward calculation that

$$(30) \qquad q[F(z)] = x_0 \log z - \sum_{n=1}^{N} \frac{x_n}{nz^n} + \sum_{m=0}^{\infty} A_m z^m,$$

where

$$(31) \qquad A_m = \sum_{n=0}^{N} [k_{mn}x_n + q_{mn}\bar{x}_n + l_{mn}y_n + p_{mn}\bar{y}_n].$$

Similarly, we can calculate

$$(32) \qquad q[\hat{F}(z)] = -y_0 \log z + \sum_{n=1}^{N} \frac{y_n}{nz^n} - \sum_{m=0}^{\infty} B_m z^m + \text{imaginary constant},$$

where

$$(33) \qquad B_m = \sum_{n=0}^{N} [k_{mn}y_n + q_{mn}\bar{y}_n + l_{mn}x_n + p_{mn}\bar{x}_n].$$

In view of the developments (30) and (32), the integrals I_1 and I_2 may be evaluated to give

$$
\begin{aligned}
I_1 &= 2\pi x_0^2 \log \rho - \pi \sum_{n=1}^{N} \frac{1}{n} |x_n|^2 \rho^{-2n} \\
&\quad + \pi \sum_{n=1}^{\infty} n|A_n|^2 \rho^{2n} + 2\pi x_0 \operatorname{Re}\{A_0\}, \\
(34) \\
I_2 &= 2\pi y_0^2 \log \rho - \pi \sum_{n=1}^{N} \frac{1}{n} |y_n|^2 \rho^{-2n} \\
&\quad + \pi \sum_{n=1}^{\infty} n|B_n|^2 \rho^{2n} + 2\pi y_0 \operatorname{Re}\{B_0\}.
\end{aligned}
$$

From (28) we conclude

$$
\begin{aligned}
(35) \quad & 2x_0 \operatorname{Re}\{A_0\} + 2y_0 \operatorname{Re}\{B_0\} + \sum_{n=1}^{\infty} n[|A_n|^2 + |B_n|^2]\rho^{2n} \\
& \qquad \leqq \sum_{n=1}^{N} \frac{1}{n} [|x_n|^2 + |y_n|^2]\rho^{-2n} + 2(x_0^2 + y_0^2) \log \rho^{-1}.
\end{aligned}
$$

This inequality holds for all $\rho < 1$, so we may pass to the limit, $\rho \uparrow 1$. We obtain an area theorem for bounded univalent Guelfer functions

$$(36) \quad 2x_0 \operatorname{Re}\{A_0\} + 2y_0 \operatorname{Re}\{B_0\} + \sum_{n=1}^{\infty} n[|A_n|^2 + |B_n|^2] \leqq \sum_{n=1}^{N} \frac{1}{n} [|x_n|^2 + |y_n|^2].$$

It now becomes useful to introduce a change of variables. Let

$$(37) \qquad \lambda_n = \tfrac{1}{2}(x_n + y_n), \qquad \mu_n = \tfrac{1}{2}(x_n - y_n)$$

and denote

$$
(38) \qquad
\begin{aligned}
a_{mn} &= k_{mn} + l_{mn}, & \alpha_{mn} &= k_{mn} - l_{mn}, \\
b_{mn} &= q_{mn} + p_{mn}, & \beta_{mn} &= q_{mn} - p_{mn}.
\end{aligned}
$$

Finally we define the new vectors

$$
C_m \equiv \tfrac{1}{2}(A_m + B_m) = \sum_{n=0}^{N} [a_{mn}\lambda_n + b_{mn}\bar{\lambda}_n],
$$

(39)

$$
D_m \equiv \tfrac{1}{2}(A_m - B_m) = \sum_{n=0}^{N} [\alpha_{mn}\mu_n + \beta_{mn}\bar{\mu}_n].
$$

Observing that

(40)
$$
x_0 A_0 + y_0 B_0 = 2\lambda_0 C_0 + 2\mu_0 D_0
$$

we find from (36) that

(41)
$$
2\lambda_0 \operatorname{Re}\{C_0\} + 2\mu_0 \operatorname{Re}\{D_0\} + \sum_{n=1}^{\infty} n[|C_n|^2 + |D_n|^2] \leq \sum_{n=1}^{N} \frac{1}{n}[|\lambda_n|^2 + |\mu_n|^2].
$$

Here we have utilized the parallelogram law for arbitrary complex numbers a, b, namely,

$$
|(a+b)/2|^2 + |(a-b)/2|^2 = \tfrac{1}{2}[|a|^2 + |b|^2].
$$

Having now separated the variables in the inequality, (41) may be split into two independent conditions. For example, taking $\mu_0 = \mu_1 = \cdots = \mu_N = 0$ we find

(42)
$$
2\lambda_0 \operatorname{Re}\{C_0\} + \sum_{n=1}^{\infty} n|C_n|^2 \leq \sum_{n=1}^{N} \frac{|\lambda_n|^2}{n}.
$$

Similarly we find

(43)
$$
2\mu_0 \operatorname{Re}\{D_0\} + \sum_{n=1}^{\infty} n|D_n|^2 \leq \sum_{n=1}^{N} \frac{|\mu_n|^2}{n}.
$$

We utilize our inequalities as follows. First consider the quantity

(44)
$$
\Phi[F] = \operatorname{Re}\left\{\sum_{m,n=0}^{N} a_{mn}\lambda_m\lambda_n\right\} + \sum_{m,n=0}^{N} b_{mn}\lambda_m\bar{\lambda}_n.
$$

From (39) we find

$$
\Phi[F] = \lambda_0 \operatorname{Re}\{C_0\} + \operatorname{Re}\left\{\sum_{m=1}^{N} \lambda_m C_m\right\}
$$

$$
\leq \lambda_0 \operatorname{Re}\{C_0\} + \left[\left(\sum_{m=1}^{N} \frac{|\lambda_m|^2}{m}\right)\left(\sum_{m=1}^{N} m|C_m|^2\right)\right]^{1/2}
$$

(45)

$$
\leq \lambda_0 \operatorname{Re}\{C_0\} + \left[\left(\sum_{m=1}^{N} \frac{|\lambda_m|^2}{m}\right)^2 - 2\lambda_0 \sum_{m=1}^{N} \frac{|\lambda_m|^2}{m} \operatorname{Re}\{C_0\}\right]^{1/2}
$$

$$
\leq \lambda_0 \operatorname{Re}\{C_0\} + \sum_{m=1}^{N} \frac{|\lambda_m|^2}{m} - \lambda_0 \operatorname{Re}\{C_0\}.
$$

Here we have used first Schwarz's inequality, then our inequality (42), and finally we have completed the square under the radical. This computation gives us

THEOREM 1. *Let $F(z) \in D_1^*$ and let a_{mn} and b_{mn} be defined by*

$$(46) \qquad \log \frac{F(z)-F(\zeta)}{[F(z)+F(\zeta)](z-\zeta)} = \sum_{m,n=0}^{\infty} a_{mn} z^m \zeta^n,$$

$$(47) \qquad \log \frac{1+F(z)\overline{F(\zeta)}}{1-F(z)\overline{F(\zeta)}} = \sum_{m,n=0}^{\infty} b_{mn} z^m \overline{\zeta}^n.$$

Then for any real number λ_0 and any N complex numbers $\lambda_1, \ldots, \lambda_N$,

$$(48) \qquad \mathrm{Re}\left\{ \sum_{m,n=0}^{N} a_{mn}\lambda_m\lambda_n \right\} + \sum_{m,n=0}^{N} b_{mn}\lambda_m\bar{\lambda}_n \leqq \sum_{m=1}^{N} \frac{|\lambda_m|^2}{m}.$$

Equality is possible only when

$$(49) \qquad \mathrm{Re}\left\{ \sum_{n=0}^{N} \left[a_{0n}\lambda_n + b_{0n}\bar{\lambda}_n \right] \right\} = 0,$$

$$(50) \qquad \sum_{n=0}^{N} \left[a_{mn}\lambda_n + b_{mn}\bar{\lambda}_n \right] = \frac{\lambda_m}{m}, \qquad m = 1, 2, \ldots, N,$$
$$= 0, \qquad m = N+1, N+2, \ldots,$$

and in this case the complement of the union of the domains $F(U)$ and $-F(U)$ with respect to the unit disc has zero area.

Similarly, we find beginning from (43) the

THEOREM 2. *Let $F(z) \in D_1^*$ and let α_{mn} and β_{mn} be defined by*

$$(51) \qquad \log \frac{F^2(z)-F^2(\zeta)}{z-\zeta} = \sum_{m,n=0}^{\infty} \alpha_{mn} z^m \zeta^n,$$

$$(52) \qquad -\log\left[1 - F^2(z)\overline{F^2(\zeta)} \right] = \sum_{m,n=0}^{\infty} \beta_{mn} z^m \overline{\zeta}^n.$$

Then for any real number μ_0 and any N complex numbers $\mu_1, \ldots, \mu_N$,

$$(53) \qquad \mathrm{Re}\left\{ \sum_{m,n=0}^{N} \alpha_{mn}\mu_m\mu_n \right\} + \sum_{m,n=0}^{N} \beta_{mn}\mu_m\bar{\mu}_n \leqq \sum_{n=1}^{N} \frac{|\mu_n|^2}{n}.$$

Equality is possible only when

$$(54) \qquad \mathrm{Re}\left\{ \sum_{n=0}^{N} \left[\alpha_{0n}\mu_n + \beta_{0n}\bar{\mu}_n \right] \right\} = 0,$$

$$(55) \qquad \sum_{n=0}^{N} \left[\alpha_{mn}\mu_n + \beta_{mn}\bar{\mu}_n \right] = \frac{\bar{\mu}_m}{m}, \qquad m = 1, \ldots, N,$$
$$= 0, \qquad m = N+1, N+2, \ldots,$$

and in this case the complement of the union of the domains $F(U)$ and $-F(U)$, taken with respect to the unit disc, has zero area.

Each of Theorems 1 and 2 gives analogous results for the unbounded class D^* by an obvious limiting procedure. If these inequalities are transformed by (8) into statements on Bieberbach-Eilenberg functions, these results coincide with those obtained by Hummel and Schiffer [4]. For definiteness we state

THEOREM 3. *Let $G(z) \in D^*$ and let the coefficients a_{mn} be defined by*

$$(56) \qquad \log \frac{G(z) - G(\zeta)}{[G(z) + G(\zeta)](z - \zeta)} = \sum_{m,n=0}^{\infty} a_{mn} z^m \zeta^n.$$

Then for λ_0 real and $\lambda_1, \ldots, \lambda_n$ any N complex numbers,

$$(57) \qquad \mathrm{Re}\left\{ \sum_{m,n=0}^{N} a_{mn} \lambda_m \lambda_n \right\} \leqq \sum_{m=1}^{N} \frac{|\lambda_m|^2}{m}.$$

To compare with Hummel-Schiffer we need only notice from (8) that

$$(58) \qquad (G(z) - G(\zeta))/(G(z) + G(\zeta)) = (E(z) - E(\zeta))/(1 - E(z)E(\zeta)).$$

If in turn $G(z)$ is related to a function $g(z) \in S$ by means of (4), then inequality (57) becomes equivalent to that of Garabedian-Schiffer.

3. **Applications.** Given a function $f(z) \in S_1$ which omits a value $\beta^2 e^{-i\varphi}$, we have seen that the $F(z)$ given by (10) is in $D^*(\beta)$. Hence the inequalities of Theorems 1 and 2 can be restated for these subclasses of S_1 functions. We find that Theorem 1 becomes equivalent to inequalities of Nehari [5, pp. 325–327] and DeTemple [1], while Theorem 2 is equivalent to the Nehari inequalities as generalized by Schiffer and Tammi [8].

Our inequalities of the preceding section may also be utilized to obtain coefficient and growth estimates for functions $F(z)$ in the class $D(\beta)$. Since the inequalities apply only to univalent functions, the following lemma for the subordination of $D(\beta)$ functions is essential.

LEMMA. *Let $F(z) \in D(\beta)$. Then there exists for $F(z)$ a univalent majorant $F^*(z) \in D^*(\beta)$. That is*

$$(59) \qquad F(z) = F^*[\omega(z)]$$

where $F^* \in D(\beta)$ and $\omega(z)$ is analytic on U with $\omega(0) = 0$, $|\omega(z)| < 1$.

The proof, which we omit, can be achieved by the obvious modifications of procedures found in either Rogosinski [6] or Guelfer [3].

A special case of inequality (48) now gives us

THEOREM 4. *Let $F(z) = \beta(1 + \alpha_1 z + \cdots + \alpha_n z^n + \cdots) \in D(\beta)$. Then*

$$(60) \qquad |\alpha_1| \leqq 2(1 - \beta^2)/(1 + \beta^2),$$

and equality holds only for the functions $w = F(z)$ *mapping* U *onto the half-disc* $\{|w| < 1, \ \mathrm{Re}\ w > 0\}$; *that is, the functions given by*

$$(61) \qquad \frac{F(z)-\beta}{1-\beta F(z)} \frac{1+\beta F(z)}{F(z)+\beta} = \varepsilon z, \qquad |\varepsilon| = 1.$$

Proof. Choose $\lambda_0 = 1$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$ in (48). Then for any function $F^*(\zeta) = \beta(1 + \alpha_1^* \zeta + \cdots) \in D^*(\beta)$ we have

$$(62) \qquad \mathrm{Re}\ \{a_{00}\} + b_{00} = \log \frac{|\alpha_1^*|}{2} + \log \frac{1+\beta^2}{1-\beta^2} \leqq 0,$$

which is equivalent to (60) in the case of univalent functions. For a general $F(z) \in D(\beta)$ we can write $F(z) = F^*(\omega(z))$ by the lemma. But then

$$(63) \qquad |\alpha_1| = |\beta^{-1} F'(0)| = |\beta^{-1} F^{*\prime}(0)| \ |\omega'(0)| \leqq |\alpha_1^*|$$

with equality only if $\omega(z) = \varepsilon z$, $|\varepsilon| = 1$.

For equality to hold in (60), we note from (49) and (50) that

$$(64) \quad C_0 = a_{00} + b_{00} = \text{imaginary}, \qquad C_m = a_{m0} + b_{m0} = 0, \qquad m = 1, 2, \ldots.$$

Subtracting (42) from (40) we then find

$$(65) \qquad q[F(z)] - q[-F(z)] = 2 \log z + 2C_0.$$

On the other hand we have from (26) that

$$(66) \qquad q[F(z)] - q[-F(z)] = 2 \log \frac{F(z)-\beta}{1-\beta F(z)} \frac{1+\beta F(z)}{F(z)+\beta} + \text{imaginary}.$$

Comparing (65) and (66) we conclude upon taking real parts that

$$(67) \qquad \log \left| \frac{F(z)-\beta}{1-\beta F(z)} \cdot \frac{1+\beta F(z)}{F(z)+\beta} \cdot \frac{1}{z} \right| \equiv 0.$$

From (67) the statements for the extremal functions follow easily. $\quad\square$

For a given $f(z) \in S(b_1)$ we can apply Theorem 4 to the function $F(z)$ given by transformation (10). Thus if $f(z) \neq d$ we obtain the well-known inequality

$$(68) \qquad |d| \geqq b_1(1 + \sqrt{(1-b_1)})^{-2},$$

with equality holding only for the function given by

$$(69) \qquad f(z)(1 + \eta f(z))^{-2} = 4|d|(1 + |d|)^{-2} z(1 + \eta z)^{-2}, \qquad \eta = \bar{d}/|d|.$$

THEOREM 5. *Let* $F(z) \in D_1$. *Then*

$$(70) \qquad \left| \frac{F'(z)}{F(z)} \right| \frac{1 + |F(z)|^2}{1 - |F(z)|^2} \leqq \frac{2}{1 - |z|^2}$$

holds for all z ∈ U. Equality, at a point ζ ∈ U, holds only for the functions

$$(71) \qquad F(z) = \frac{\overline{F_1(-\zeta)}}{|F_1(-\zeta)|} \, F_1\!\left(\frac{z-\zeta}{1-\overline{\zeta}z}\right)$$

where $F_1(z)$ has the form given by (61).

Proof. Let $F(z) \in D_1$, $\zeta \in U$, and form

$$(72) \qquad F_1(z) = \frac{\overline{F(\zeta)}}{|F(\zeta)|} \, F\!\left(\frac{z+\zeta}{1+\overline{\zeta}z}\right).$$

Then $F_1(z) \in D(|F(\zeta)|)$ and

$$(73) \qquad F_1'(0) = \frac{\overline{F(\zeta)}}{|F(\zeta)|} \, F'(\zeta)(1-|\zeta|^2).$$

The result now follows by application of Theorem 4 to $F_1(z)$. □

THEOREM 6. *Let $F(z) \in D(\beta)$. Then*

$$(74) \qquad \left| \log \frac{F(z)}{\beta} \frac{1-\beta^2}{1-F^2(z)} \right| \leqq \log \frac{1+|z|}{1-|z|}$$

and in particular

$$(75) \qquad \frac{1-|z|}{1+|z|} \leqq \frac{\beta}{1-\beta^2} \left| \frac{1}{F(z)} - F(z) \right| \leqq \frac{1+|z|}{1-|z|},$$

$$(76) \qquad \left| \arg\left(\frac{1}{F(z)} - F(z)\right) \right| \leqq \log \frac{1+|z|}{1-|z|}.$$

Proof. Integrate $(dF/F)((1+F^2)/(1-F^2))$ over the line segment $\langle 0, z \rangle$, and use inequality (70). □

For $F(z) \in D(\beta)$ we have seen $G(z) = \beta^{-1}F(z) \in D$. If we replace $F(z)$ by $\beta G(z)$ in our inequalities of this section, then results of Guelfer for the class D are obtained upon sending $\beta \to 0$.

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